Intergrerend project dynamische systemen

4 April 2014

university of groningen

Exam Problem Sheet

The exam consists of 4 problems. You may answer in Dutch or in English. You have 90 minutes to answer the questions. Give brief but precise answers. You can achieve 50 points in total which includes a bonus of 5 points.

1. [**3+5+2 Points.**]
   Consider the one-dimensional growth model

$$x' = ax - h \sin t,$$

   where a and h are positive real constants.

   (a) Determine the general solution and show that the system has exactly one periodic solution.

   (b) Determine the Poincaré map of the system and show that it has exactly one fixed point which is always a source.

   (c) Use the results in part (b) to sketch the solution curves of the original time continuous system in the (t, x)-plane.

2. [**3+3+4+2 Points.**]
   Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function. Then the system

$$X' = -\nabla V(X) \tag{1}$$

   is called a *gradient system* (here $\nabla V(X) = (\frac{\partial}{\partial x_1} V(X), \dots, \frac{\partial}{\partial x_n} V(X))$ with $X = (x_1, \dots, x_n) \in \mathbb{R}^n$). It is clear that the equilibrium points of a gradient system are given by the critical points of V.

   (a) Show that if X is not an equilibrium point, then V is strictly decreasing along the solution curve through X.

   (b) State the definition of asymptotic stability for the equilibrium point of a time continuous system.

   (c) Show that if $X^\star$ is an isolated minimum of V then $X^\star$ is asymptotically stable. What can you say about the basin of attraction of $X^\star$ in this case?

   (d) What are the conditions on V at an equilibrium point $X^\star$ to conclude stability from the linearization of the gradient system at $X^\star$?

—— **please turn over** ——

3. [**2+4+5 Points.**]

   Consider the planar system

$$X' = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} X\,.$$

   (a) Determine the canonical form of the system.

   (b) Determine the phase portraits of the original system and its canonical form.

   (c) Determine a conjugacy relating the flows of the system to its canonical form.

4. [**3+9 Points.**]

   (a) State the definition of chaos for a discrete time system.

   (b) Argue that the doubling map

$$d : [0,1] \to [0,1], \quad x \mapsto 2x \bmod 1$$

   is chaotic.

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- Solutions -

1. (a) $x' = ax - h\sin t$

$x_h' = a x_h \Rightarrow x_h(t) = K e^{at}$, $K \in \mathbb{R}$

particular solution:

ansatz $x_p(t) = A\cos t + B\sin t$

$\Rightarrow x_p'(t) = -A\sin t + B\cos t$

$\overset{!}{=} aA\cos t + aB\sin t - h\sin t$

equating coefficients of sin and cos gives:

$$-A = aB - h$$
$$B = aA$$

$$\Leftrightarrow \quad -A = a^2 A - h$$
$$B = aA$$

$$\Leftrightarrow \quad A = \frac{h}{1+a^2}$$
$$B = \frac{ah}{1+a^2}$$

$\Rightarrow$ general solution

$$x(t) = x_h(t) + x_p(t)$$
$$= K e^{at} + \frac{h}{1+a^2}\cos t + \frac{ah}{1+a^2}\sin t$$

the unique periodic solution as $K = 0$

(b) $x(0) = x_0$

$$\Rightarrow \quad x_0 = K + \frac{h}{1+a^2}$$

$$\Rightarrow \quad K = x_0 - \frac{h}{1+a^2}$$

$$\Rightarrow \quad x(t) = \left(x_0 - \frac{h}{1+a^2}\right)e^{at} + \frac{h}{1+a^2}(\cos t + a\sin t)$$

$= \phi_t(x_0)$ where ϕ is the flow system is 2π-periodic. So the Poincaré map is given

by $$P(x) = \phi_{2\pi}(x) = \left(x - \frac{h}{1+a^2}\right)e^{a2\pi} + \frac{h}{1+a^2}$$

fixed point: $P(x) = x$

$$\Leftrightarrow \quad (e^{2\pi a} - 1)x = \frac{h}{1+a^2}e^{a2\pi} - \frac{h}{1+a^2}$$

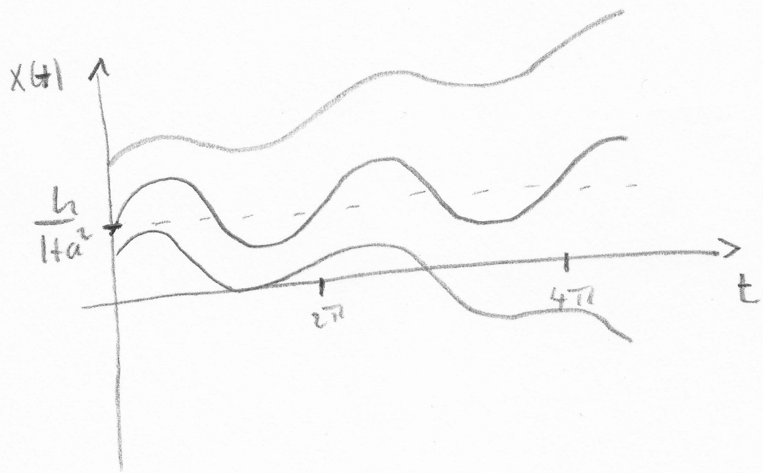
$$\Leftrightarrow \quad x = \frac{h}{1+a^2}$$

stability:

$$P'\left(\frac{h}{1+a^2}\right) = e^{a2\pi} > 1 \quad \text{since } a > 0$$

$\Rightarrow$ $x = \frac{h}{1+a^2}$ is unique fixed point which is a source

(c)



2\. (a) Let $X(t)$ be a non-constant solution.

Then $\frac{d}{dt} V(X(t)) = \nabla V(X(t)) \cdot X'(t) = -\nabla V(X(t)) \cdot \nabla V(X(t))$

$= -|\nabla V(X(t))|^2 < 0$ as $X'(t) = \nabla V(X(t)) \neq 0$

(b) Let X_0 be an equilibrium point.

Then X_0 is called asymptotically stable if

X_0 is (Lyapunov) stable, i.e. for each neighbourhood θ of X_0 there exists a neighbourhood θ_1 of X_0 such that for all initial conditions $X(0) \in \theta_1$ it holds that $X(t) \in \theta$ for all $t \geq 0$, and moreover θ_1 can be chosen such that for all $X(0) \in \theta_1$,

$X(t) \to X_0$.

(c) Choose V as a Lyapunov function.

If X^* is an isolated minimum then $\nabla V(X^*) = 0$ and there exists a neighbourhood θ such that $\nabla V(X) \neq 0$ for $X \in \theta \setminus \{X^*\}$.

Together with the result from part (a) we see that V is a strict Lyapunov function on θ and hence by the Lyapunov Th^{m} we can infer asymptotic stability of X^*. Moreover θ belongs then to the basin of attraction of X^*.

In particular we can choose θ to be the open region enclosed by a level set $\{X \in \mathbb{R}^n : V(X) < c\}$.

(or more precisely the connected component of such a region which contains X^*)

(d) The linearization at X^* is given by

$X' = HX$ where H is the Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 V}{\partial x_1^2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & & \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n^2} \end{pmatrix} \text{ at } X^*.$$

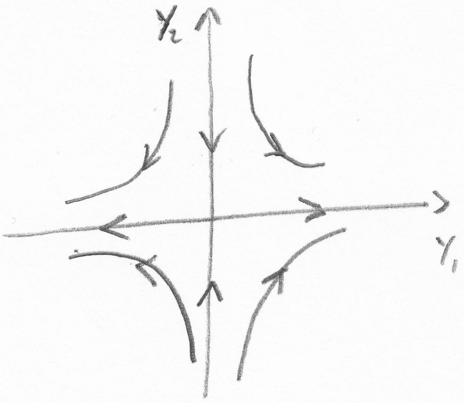
Since H is symmetric its eigenvalues are real. Hence H is hyperbolic if H has only non-vanishing eigenvalues.

Then: if all eigenvalues are negative then X^* is asymptotically stable. If one eigenvalue is positive then X^* is not stable.

3. (a) The matrix defining the linear system is upper triangular. The eigenvalues can hence be read off from the diagonal and are 2 and -1.
The canonical form is

$$y' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} y$$

(b) phase portrait canonical form:



phase portrait original system:

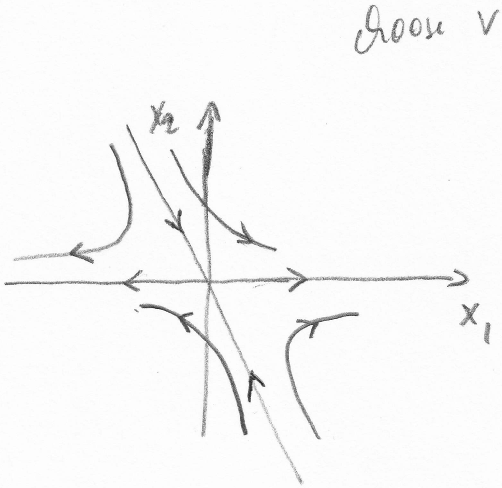
determine eigenvectors:

$$2: \quad \begin{pmatrix} 2-2 & 1 \\ 0 & -1-2 \end{pmatrix} u = 0 \iff \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

choose $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$-1: \quad \begin{pmatrix} 2-(-1) & 1 \\ 0 & (-1)-(-1) \end{pmatrix} v = 0 \iff \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

choose $v = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$



(c) We have

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

Let $T = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$

The flow of the original system is

$$\phi_t = \exp\left[\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} t\right]$$

The flow of the canonical system is

$$\phi_t^c = \exp\left[\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} t\right]$$

The flows are conjugate by T:

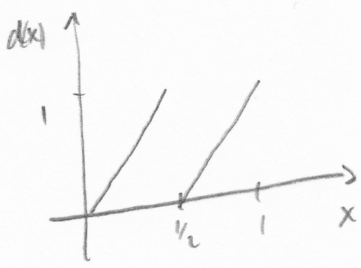
$$\phi_t^c = T^{-1} \phi_t T$$

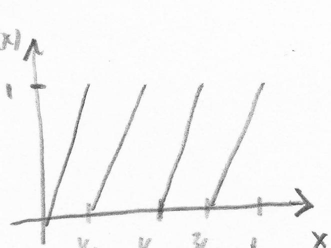
4. (a) Consider $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \dots$

The system is chaotic if

1. periodic orbits are dense, i.e. for all x_0 and each open neighbourhood U of x_0 there exists a periodic point contained in U
2. the system is transitive, i.e. for all open sets U and V there exists $n \in \mathbb{N}$ such that
$$f^n(U) \cap V \neq \emptyset$$
3. the system has sensitive dependence on initial conditions, i.e. $\exists \beta > 0$ such that for all x and open neighbourhood U of x there is $y \in U$ and $n > 0$ such that
$$d(f^n(x), f^n(y)) > \beta$$

(b) consider the graphs of d and its iterates





d^n maps 2^n intervals

$$I_k^n := \left[(k-1)\left(\tfrac{1}{2}\right)^n, k\left(\tfrac{1}{2}\right)^n\right] \quad , \quad k = 1, \dots, 2^n$$

surjectively to the interval $[0,1)$

We have $[0,1) = \bigcup_{k=1}^{2^n} I_k^n$ and length of I_k^n equal to $\left(\frac{1}{2}\right)^n$

1. Each interval I_k^n contains a periodic point of period n.

$\Rightarrow$ periodic orbits are dense.

2. Let $U, V \subset [0,1]$ open

$\Rightarrow \exists\, n, k$ such that $I_k^n \subset U$

$\Rightarrow d^n(I_k^n) = [0,1] \cap V \neq \emptyset$

$\Rightarrow d$ is transitive

3. d has sensitive dependence on initial conditions.

Choose $\beta = 2$. $\Rightarrow$

$|d(x) - d(y)| \geq \beta |x - y|$ for all $x, y \in [0, \frac{1}{2}]$ and $x, y \in [\frac{1}{2}, 1]$

The rest is clear.